Similar derivatives

Teacher notes

Why use this resource?

This resource offers students the opportunity to use similar triangles to construct geometrical arguments which lead to the derivatives of $\tan\theta$, $\sec\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$. In doing this, students may develop a deeper understanding of the derivatives of these functions, and also reinforce their understanding of the connections between the unit circle diagram, trigonometric functions and identities such as $\sec^2\theta \equiv 1 + \tan^2\theta$.

Possible approach

(Please note that the diagram in the printable version of the problem has $\delta\theta \neq 0$. If you choose to give this page to the students, it may affect how you introduce the problem.)

Students could be asked to think about why the two points have the coordinates given before trying to explain this to a neighbour, or this could form a class plenary of ideas to bring out relationships within the diagram.

Once students have had a chance to discuss these points, slowly increase $\delta\theta$ to its maximum value and ask the questions in the resource.

As well as noticing that the blue triangles get very small as $\delta\theta \to 0$, students should also think about the geometric implications of the blue angle approaching the same size as the red angle. There are two diagrams in the Suggestion section that may help with this. In particular, it would be helpful to bring out in discussion that the blue triangles are similar to triangle OQT, which tends to the red triangle OQP as $\delta\theta$ tends to 0.

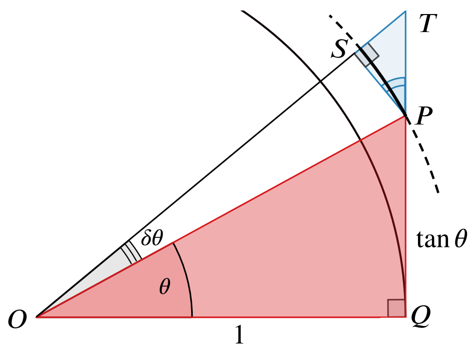

Image from suggestion section

Key questions

- What can we say about the similar triangles?
- What do you see as $\delta\theta \to 0$?
- How can you work out the length of a circular arc?
- Why is it important to measure θ and $\delta\theta$ in radians?
- Which lengths show the increase in $\tan\theta$ or $\sec\theta$ as θ increases to $\theta + \delta\theta$?

To start thinking more generally about the derivatives obtained this way,

- Do your arguments hold if θ is not acute?
- Think about the graphs of these functions. Do the gradient functions you've obtained make sense?

Possible support

The diagrams in the suggestion tab can be used to focus students' attention on similar triangles and labelling lengths in terms of θ.

For further support, the labelled diagram in the suggestion could be printed separately. Students could be asked to label as many lengths in it as they can in terms of θ and $\delta\theta$ and describe them in words. In particular, this may help to support discussion of the fact that PT and ST show the increase in $\tan\theta$ and $\sec\theta$ respectively as θ increases to $\theta + \delta\theta$.

The solution discusses differentiating $\tan\theta$ in some detail. This could be used to provide scaffolding for this part of the problem and then students could be asked to construct similar arguments for $\sec\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$.

Possible extension

Ask students to think about how the arguments may or may not hold if θ is not acute.

Can students construct a triangle or triangles that would give a similar argument for the derivatives of $\sin\theta$ and $\cos\theta$?

As a way of interpreting the results from this resource, students could be asked to consider the derivatives of all six trigonometric functions in the following forms. They may notice how the derivatives appear to come in pairs, with the radii of circles arising as scale factors.

Function	**Derivative**
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec\theta \times \sec\theta$
$\sec\theta$	$\tan\theta \times \sec\theta$
$\cot\theta$	$-\operatorname{cosec}\theta \times \operatorname{cosec}\theta$
$\operatorname{cosec}\theta$	$-\cot\theta \times \operatorname{cosec}\theta$