Determinant

The *determinant* of a matrix is a number which tells us something about the properties of the matrix. For example, if the matrix represents a 2-dimensional linear transformation, then the determinant will tell us the ratio by which areas are scaled, while for a 3-dimensional linear transformation, it will give the volume ratio.

Often the determinant of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{or} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For this 2×2-matrix, the determinant is $\det A = ad - bc$. In this case the determinant is the area of a parallelogram with sides given by the vectors

$$\begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ d \end{pmatrix},$$

which is the image of the unit square with sides $\mathbf{i} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{j} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

For a 3×3-matrix, we can find the determinant as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

(This is called the *Laplace expansion* or the *cofactor expansion* of the determinant. This generalises to $n \times n$-matrices.) In this case, the determinant is the volume of the parallelepiped with edges given by the vectors

$$\begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c \\ f \\ i \end{pmatrix},$$

which is the image of the unit cube with sides $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.