Telescoping a series

Telescoping is a method for finding the sum of certain series.

If we have a sequence a_0, a_1, a_2, ..., and we can find a function $f(n)$ such that $a_n = f(n+1) - f(n)$ for every n, then

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} f(i+1) - f(i) = f(n+1) - f(0)$$

(because all the other terms cancel).

But finding such a function explicitly may be difficult.

As an example, for the sequence $a_n = n$, we note that $(n+1)^2 - n^2 = 2n+1$, and $(n+1) - n = 1$, so if we take $f(n) = \frac{1}{2}(n^2 - n)$, we will have

$$\begin{aligned} f(n+1) - f(n) &= \tfrac{1}{2}\big((n+1)^2 - (n+1)\big) - \tfrac{1}{2}(n^2 - n) \\ &= \tfrac{1}{2}(n^2 + n) - \tfrac{1}{2}(n^2 - n) \\ &= n \\ &= a_n \end{aligned}$$

so that

$$\begin{aligned} \sum_{i=0}^{n} i &= f(n+1) - f(0) \\ &= \tfrac{1}{2}\big((n+1)^2 - (n+1)\big) - \tfrac{1}{2}(0^2 - 0) \\ &= \tfrac{1}{2}n(n+1). \end{aligned}$$