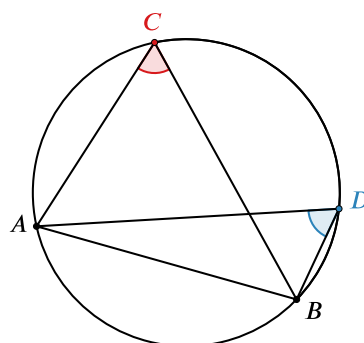


Consider the statement that ‘the angles of a planar triangle add up to  $180^\circ$ ’. Perhaps the most easily overlooked word is ‘a’, but it is important because it refers not to a specific triangle but to all possible triangles in the plane. The statement is saying that no matter how a triangle varies, the sum of the angles is invariant. Another example of invariance is that angles at the circumference on the same side of a chord are equal, so in the diagram below angles  $ACB$  and  $ADB$  are always equal provided  $C$  and  $D$  are on the same side of the chord  $AB$ .

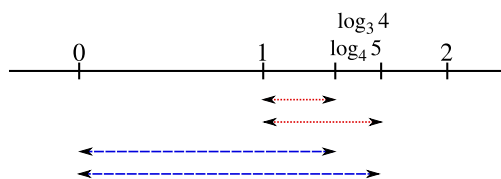


Trigonometry ‘works’ because of the invariance of the ratio of corresponding sides in similar triangles, so holding one angle fixed at  $90^\circ$  allows those ratios to characterise or provide a measure of the other two angles. Notice that this makes use of the fact that in a right-angled triangle, the sum of the other two angles is invariant, so each right-angled triangle is characterised by one of its ‘other’ angles. [Equation or identity? \(I\)](#) explores how we can distinguish between trigonometric relationships that hold for all triangles and those that may only hold for some or none. Another example of invariance in trigonometry is the [sine rule](#) or law of sines. In [Sine-ing the way](#), we can see how this relationship arises and how the invariant ratio is related to the [circumcircle](#) of the triangle.

Many if not most mathematical results can be seen as identification of some relationship or relationships that are invariant under some specified domain of change. What that domain is matters. For example, the rule for differentiating  $x^n$  applies to any index  $n$  as long as  $n$  is constant; it does not apply to  $x^x$ . Similarly the sum to infinity of a geometric progression with first term  $a$  and constant ratio  $r$  is only  $\frac{a}{1-r}$  as long as  $|r|$  is less than 1, and  $\ln x^2 = 2 \ln x$  only if  $x > 0$ . Whenever you encounter a result that you want to use in some reasoning or to resolve a problem, make sure that any necessary [constraints](#) are being met, or in other words, that the invariance you want to use is valid over the range of change in which you are interested.

Invariance underlies the algebraic manipulation we use to solve equations and inequalities. Think of how you’ve recently used invariance in the following forms: the value of a fraction is invariant under multiplication of the numerator and denominator by any non-zero number (integer, rational, real or complex!); the difference of two numbers is invariant under adding the same number to both (which can be used to surprising effect in [Logarithm lineup](#)); the

product of two numbers is invariant under multiplying one and dividing the other by the same non-zero number; the sum of two numbers is invariant under adding a number to one and subtracting the same number from the other.

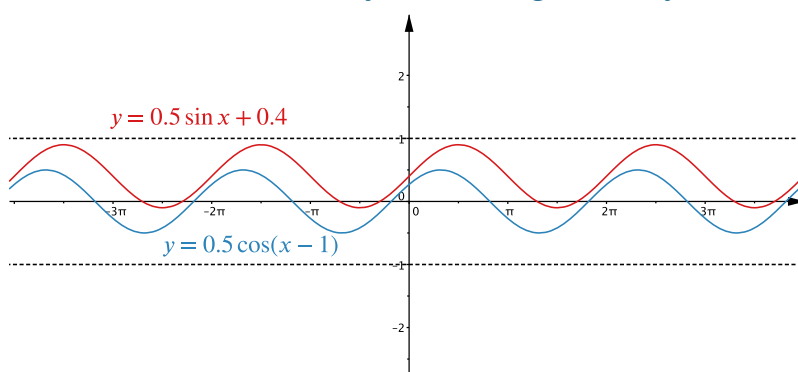


### From Logarithm lineup

As well as thinking of these results algebraically, they can be explored graphically by considering points on graphs such as  $xy = c$  or  $y + x = c$ . Moving to more general straight line graphs, we can see that the equation of a straight line is invariant under multiplying all the coefficients (of  $x$ , of  $y$  and the constant) by the same non-zero number.

**Transformers** encourages thinking about **transformations** when comparing similar graphs, which brings us to the question of which features of lines and curves remain invariant if we transform the plane. We can use this to help us sketch graphs of functions. Imagine, for example, sketching the graph of  $y = \ln(4 - x)$ . You may think of reflecting  $y = \ln x$ , but in what line? How could features such as the asymptote and intercept be used to help sketch the graph?

We might also think about how transforming the plane affects families of curves. For example, circles remain circles when the plane is translated or rotated, or scaled, provided the plane is scaled in the same way in all directions, which is an underlying idea in **Teddy bear**. Ellipses (which include circles) remain ellipses when the plane is translated, rotated, scaled, or sheared (different scaling in different directions). But what happens to parabolas under different transformations? This can be explored in **Which parabola?** and also in **It's a matter of perspective**. Thinking about graphs of periodic functions, we see that the period is invariant under scaling of the  $y$ -axis, and under translation. This can be used to generate examples that meet the constraints in **Can you find...trigonometry edition**.

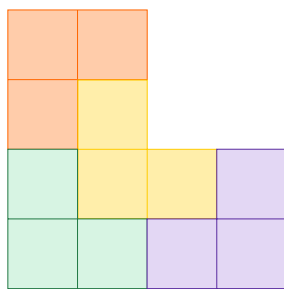


### From Can you find...trigonometry edition

A frequently used example of invariance is the fact that we can always factorise the difference of two squares. This fact remains invariant under changes in what the 'squares'

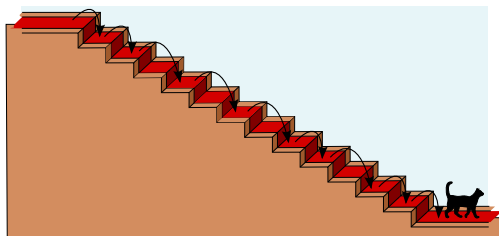
are: they could be actual numbers or algebraic expressions. See, for example, review questions [Can we sum the first  \$2n\$  terms of  \$1^2 - 3^2 + 5^2 - 7^2 + \dots\$ ?](#) and [Can we factorise  \$\(x + y\)^6 - \(x - y\)^6\$  completely?](#) and the resource  [\$n^5 - n\$](#) , which asks What is the biggest number that divides all numbers in the sequence  $1^5 - 1, 2^5 - 2, 3^5 - 3, \dots, n^5 - n, \dots$ ? Noticing a difference of two squares often provides a way in to solving a bigger problem.

If an object such as  $n^5 - n$  has a property which holds for any positive integer  $n$ , then this property remains invariant as we pass from  $n$  to  $n + 1$ . This is a central idea in proof by induction, and offers an alternative way to prove the result mentioned above about the highest factor of  $n^5 - n$  for positive integers  $n$ . When trying to prove a result by induction, we look for ways in which objects are related to smaller or previous examples, such as how terms in a sequence are related to previous terms. [Triominoes](#) offers a visual way to see how a proof for each case can build on the previous case.



### From Triominoes

A formula that defines a sequence by telling you how to get to the next term of the sequence from previous ones is called a [recurrence relation](#) and is an invariant relationship. For example the triangular numbers can be defined by  $a_{n+1} = a_n + n + 1$  and  $a_1 = 1$  as well as by the closed formula  $a_n = \frac{n}{2}(n + 1)$ . Similarly, the Fibonacci numbers  $F(n)$  are defined by the recurrence relation  $F(n) = F(n - 1) + F(n - 2)$  and  $F(1) = F(2) = 1$ . Establishing invariant relationships between terms in a sequence can help us to understand how the sequence behaves and may help us to find a closed formula, or understand the situation from which the sequence was generated. [One step, two step](#) offers an opportunity to make sense of how a situation can lead to recursive relationships.



### From One step, two step

Invariance is one of the ideas that we have chosen to highlight in our [pervasive ideas](#).